Direct methods for solving linear systems:

Linear systems of equations are associated with many problems in engineering and science, as well as with applications of mathematics to the social sciences.

Direct techniques are considered to solve the linear system:

$$
\begin{aligned}
&a_{1.1}\, x_1 + a_{1.2}\, x_2 + \cdots + a_{1.n}\, x_n = b_1 \\
&a_{2.1}\, x_1 + a_{2.2}\, x_2 + \cdots + a_{2.n}\, x_n = b_2 \\
&\vdots \\
&a_{n.1}\, x_1 + a_{n.2}\, x_2 + \cdots + a_{n.n}\, x_n = b_n
\end{aligned}
$$

for $x_1, x_2, \ldots\ldots, x_n$ given the $a_{i.j}$ for each (i,j=1,2,…,n) direct techniques are methods that give an answer in a fixed number of steps subject only to rounding errors.

Linear systems of equations:

Example(1): Consider the four equations

$$x_1 + x_2 + 3x_4 = 4 \rightarrow (1)$$

$$2x_1 + x_2 - x_3 + x_4 = 1 \rightarrow (2)$$

$$3x_1 - x_2 - x_3 + 2x_4 = -3 \rightarrow (3)$$

$$-x_1 + 2x_2 + 3x_3 - x_4 = 4 \rightarrow (4)$$

Will be solved for the unknowns $x_1, x_2, x_3,$ and x_4 the first step is to use equation (1) to eliminate the unknown x_1 from equations. (2),(3), and (4) by performing (2)-2(1),(3)-3(1), and (4)+(1) the resulting system is

$$x_1 + x_2 + 3x_4 = 4 \rightarrow (1^\backslash)$$

$$-x_2 - x_3 - 5x_4 = -7 \rightarrow (2^\backslash)$$

$$-4x_2 - x_3 - 7x_4 = -15 \rightarrow (3^\backslash)$$

$$3x_2 + 3x_3 + 2x_4 = 8 \rightarrow (4^\backslash)$$

Where the new equations are labeled $(1^\backslash),(2^\backslash),(3^\backslash),(4^\backslash)$ in this system $(2^\backslash)$ is used to eliminate x_2 from $(3^\backslash), \textit{and}\ (4^\backslash)$ by the operations $(3^\backslash)-4(2^\backslash)\ \textit{and}\ (4^\backslash)+3(2^\backslash)$ resulting in the system

$$x_1 + x_2 + 3x_4 = 4$$

$$-x_2 - x_3 - 5x_4 = -7$$

$$3x_3 + 13x_4 = 13$$

$$-13x_4 = -13$$

The system now in reduced form and can easily be solved for the unknowns by a backward substitution process noting that $x_4 = 1$

the solution is therefore

$$x_1 = -1 \,,\, x_2 = 2 \,,\, x_3 = 0 \,, and \;\; x_4 = 1$$

Definition:

An (mxn) matrix is a rectangular array of elements with n rows and m columns in which not only is the value of an element important but also its position in the array.

$$A = \left(a_{i,j}\right) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

An (n+1)* n matrix can be used to represent the linear system.

$$a_{1.1}\ x_1 + a_{1.2}\ x_2 + \cdots + a_{1.n}\ x_n = b_1$$
$$a_{2.1}\ x_1 + a_{2.2}\ x_2 + \cdots + a_{2.n}\ x_n = b_2$$
$$\vdots$$
$$a_{n.1}\ x_1 + a_{n.2}\ x_2 + \cdots + a_{n.n}\ x_n = b_n$$

by first constructing $A = \begin{bmatrix} a_{1.1} & a_{1.2} & \cdots & a_{1.n} \\ a_{2.1} & a_{2.2} & \cdots & a_{2.n} \\ \vdots & & & \vdots \\ a_{n.1} & a_{n.1} & \cdots & a_{n.n} \end{bmatrix} and \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

and then combining these matrices to form the augmented matrix:

$$A = \left[\begin{array}{cccc||c} a_{1.1} & a_{1.2} & \cdots & a_{1.n} & b_1 \\ a_{2.1} & a_{2.2} & \cdots & a_{2.n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{n.1} & a_{n.1} & \cdots & a_{n.n} & b_2 \end{array}\right]$$

Where the broken line is used to separate the coefficients of the unknowns from the values on the right-hand side of the equations.

Now, repeating the operations involved in Example(1): In considering first the augmented matrix associated with the system

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 4 \\ 2 & 1 & -1 & 1 & 1 \\ 3 & -1 & -1 & 2 & -3 \\ -1 & 2 & 3 & -1 & 4 \end{array}\right] \rightarrow \qquad *$$

Performing the operations associated with (2) -2(1), (3) -3(1) and (4) +(1) is accomplished by manipulating the respective rows of the augmented matrix * which becomes

the matrix

$$\left[\begin{array}{cccc||c} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & -4 & -1 & -7 & -15 \\ 0 & 3 & 3 & 2 & 8 \end{array}\right]$$

Performing the final manipulation results in the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & 5 & -7 \\ 0 & 0 & 3 & 13 & 13 \\ 0 & 0 & 0 & -13 & -13 \end{array}\right]$$

This matrix can now be transformed into its correspond linear system and solutions for (x_1, x_2, x_3, x_4) obtained

the procedure involved in this process is called

(Gaussian elimination with backward substitution.)

1)Gaussian Elimination:

The general form applied to the linear system

$$a_{1.1}\ x_1 + a_{1.2}\ x_2 + \cdots + a_{1.n}\ x_n = b_1$$

$$a_{2.1}\ x_1 + a_{2.2}\ x_2 + \cdots + a_{2.n}\ x_n = b_2$$

$$\vdots$$

$$a_{n.1}\ x_1 + a_{n.2}\ x_2 + \cdots + a_{n.n}\ x_n = b_n$$

$$A = \left[A, b\right] = \left[\begin{array}{cccc|c} a_{1.1} & a_{1.2} & \cdots & a_{1.n} & a_{1.n+1} \\ a_{2.1} & a_{2.2} & \cdots & a_{2.n} & a_{2.n+1} \\ \vdots & & & \vdots & \vdots \\ a_{n.1} & a_{n.1} & \cdots & a_{n.n} & a_{n.n+1} \end{array}\right]$$

<u>the resulting matrix will be</u>

$$\tilde{\tilde{A}} = \left[\begin{array}{cccc|c} a_{1.1} & a_{1.2} & \cdots & a_{1.n} & a_{1.n+1} \\ 0 & a_{2.2} & \cdots & a_{2.n} & a_{2.n+1} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{n.n} & a_{n.n+1} \end{array}\right]$$

The backward substitution can be performed solving the (n^th^) equation for (X_n)

gives

$$\left(X_n = \frac{a_{n,n+1}}{a_{n,n}} \right)$$

solving the (n-1)st equation for (x_{n-1}) and using (x_n) yields

$$\left[x_{n-1} = \frac{(a_{n-1,n+1} + a_{n-1,n} x_n)}{a_{n-1,n-1}} \right]$$

and continuing this process we obtain

$$x_i = \left(\frac{a_{i,n+1} - a_{i.n} x_n - a_{i,n-1} \ x_{n-1} - \cdots - a_{i,i+1} x_{i+1}}{a_{i,i}} \right) = \frac{\left(a_{i,n+1} \sum_{j=i+1}^{n} a_{i,j} \ x_j \right)}{a_{i,i}}$$

for each $(i = n-1, n-2, \cdots, 2, 1)$

Example: solve the linear system using the elimination method:

$$x_1 + x_2 + x_3 + x_4 = 7$$
$$x_1 + x_2 + 2\,x_4 = 8$$
$$2x_1 + 2x_2 + 3x_3 = 10$$
$$-x_1 - x_2 - 2x_3 + 2\,x_4 = 0$$

$$A = [A, b] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 1 & 1 & 0 & 2 & 8 \\ 2 & 2 & 3 & 0 & 10 \\ -1 & -1 & -2 & 2 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -2 & -4 \\ 0 & 0 & -1 & 3 & 7 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

Performing backward substitution

$x_4 = 3, x_3 - 2x_4 = -4, \quad \therefore x_3 = 2 \quad (x_2)$ arbitrary and

$x_1 = -2 - x_2$ there is no unique solution.

Example: solve the linear system using the elimination method:

$$x_1 - x_2 + 3x_3 = 2$$

$$3x_1 - 3x_2 + x_3 = -1$$

$$x_1 + x_2 = 3$$

Solution: Row interchange necessary:

$$\therefore \tilde{\tilde{A}} = [A, b] = \left[\begin{array}{ccc||c} 1 & -1 & 3 & 2 \\ 1 & 1 & 0 & 3 \\ 3 & -3 & 1 & -1 \end{array}\right] = \left[\begin{array}{ccc||c} 1 & -1 & 3 & 2 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & -8 & -7 \end{array}\right] \qquad x_3 = \frac{7}{8} = 0.875.$$

$$2x_2 - 3x_3 = 1 \ , \ \therefore x_2 = 1.18125.$$

$$x_1 - x_2 + 3x_3 = 2 \qquad \therefore x_1 = 1.1875.$$

Note: the difficulty of Gaussian method is that some time you have to interchange rows and some times you will not have a unique answer to the solution.

2)Direct Factorization Methods:

In the Gaussian elimination method, the system was reduced to a triangular form and then solved by back substitution. It is much easier to solve triangular systems, let us exploit this idea and assume that a given (N*N) matrix (A) can be written as a product of two matrices (L) and (U) so that

$$A = LU \rightarrow (1)$$

where L is an NxN lower- triangular matrix and U is an (NxN) upper-triangular matrix. The factorization in (1) is called an LU decomposition of A. then Ax=b is equivalent

To $LUx = b$. further $L(Ux) = b$ decomposes into two triangular systems $Ux = y$ and $Ly = b$ both systems are triangular and therefore easy to solve. What we need is a procedure to generate factorization.

Theorem:

Let A be an NxN matrix. If det $\left(A_k\right) \neq 0$ for K=1,2,…,N -1 (Gaussian elimination can be carried out without row interchanges). Then there exists a unique lower-triangular matrix L with $l_{1.1} = 1$

and a unique upper-triangular matrix U such that $LU = A$ so, far we used $l_{1.1} = 1$ in L. if we require L and U to be lower and upper-triangular matrices then we can select L and U in many ways. Let us consider a 4x4 matrix

$$A = \begin{bmatrix} a_{1.1} & a_{1.2} & a_{1.3} & a_{1.4} \\ a_{2.1} & a_{2.2} & a_{2.3} & a_{2.4} \\ a_{3.1} & a_{3.2} & a_{3.3} & a_{3.4} \\ a_{4.1} & a_{4.2} & a_{4.3} & a_{4.4} \end{bmatrix} \rightarrow \quad (2)$$

Then the factors of A are given by (assuming it is factorable)

$$L = \begin{bmatrix} l_{1.1} & 0 & 0 & 0 \\ l_{2.1} & l_{2.2} & 0 & 0 \\ l_{3.1} & l_{3.2} & l_{3.3} & 0 \\ l_{4.1} & l_{4.2} & l_{4.3} & l_{4.4} \end{bmatrix} \rightarrow (3)$$

$$and \quad U=\begin{bmatrix} u_{1.1} & u_{1.2} & u_{1.3} & u_{1.4} \\ 0 & u_{2.2} & u_{2.3} & u_{2.4} \\ 0 & 0 & u_{3.3} & u_{3.4} \\ 0 & 0 & 0 & u_{4.4} \end{bmatrix} \rightarrow (4)$$

We have 16 Known elements for A in equation (2) and 20 unknowns for L and U in equations (3) and (4). For a unique solution of a system having 20 unknowns, we need 20 equations while we have 16 equations since $LU = A$. So we specify four additional conditions on the unknowns in the **following well-known ways:**

1) Crouts method: let $u_{1.1} = u_{2.2} = u_{3.3} = u_{4.4} = 1$

2)Doolittle's method: let $l_{1.1} = l_{2.2} = l_{3.3} = l_{4.4} = 1$

Now, let us consider crouts method. In this method we want

$$\begin{bmatrix} a_{1.1} & a_{1.2} & a_{1.3} & a_{1.4} \\ a_{2.1} & a_{2.2} & a_{2.3} & a_{2.4} \\ a_{3.1} & a_{3.2} & a_{3.3} & a_{3.4} \\ a_{4.1} & a_{4.2} & a_{4.3} & a_{4.4} \end{bmatrix} = \begin{bmatrix} l_{1.1} & 0 & 0 & 0 \\ l_{2.1} & l_{2.2} & 0 & 0 \\ l_{3.1} & l_{3.2} & l_{3.3} & 0 \\ l_{4.1} & l_{4.2} & l_{4.3} & l_{4.4} \end{bmatrix} \begin{bmatrix} 1 & u_{1.2} & u_{1.3} & u_{1.4} \\ 0 & 1 & u_{2.3} & u_{2.4} \\ 0 & 0 & 1 & u_{3.4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{1.1} & l_{1.1}u_{1.2} & l_{1.1}u_{1.3} & l_{1.1}u_{1.4} \\ l_{2.1} & l_{2.1}u_{1.2} + l_{2.2} & l_{2.1}u_{1.3} + l_{2.2}u_{2.3} & l_{2.1}u_{1.4} + l_{2.2}u_{2.4} \\ L_{3.1} & l_{3.1}u_{1.2} + l_{3.2} & l_{3.1}u_{1.3} + l_{3.2}u_{2.3} + l_{3.3} & l_{3.1}u_{1.4} + l_{3.2}u_{2.4} + l_{3.3}u_{3.4} \\ L_{4.1} & l_{4.1}u_{1.2} + l_{4.2} & l_{4.1}u_{1.3} + l_{4.2}u_{2.3} + l_{4.3} & l_{4.1}u_{1.4} + l_{4.2}u_{2.4} + l_{4.3}u_{3.4} + l_{4.4} \end{bmatrix} \ldots\ldots(5)$$

Comparing each element of the first column and first row equation (5) we get :

$$l_{i.1} = a_{i.1} \quad for \quad i = 1,2,3 \; and \; 4 \qquad u_{1.j} = \frac{a_{1.j}}{l_{1.1}} \quad for \;\; j = 2,3 \quad and \; 4$$

Comparing the last three elements of the second column

$$l_{i.2} = -l_{i.1}u_{1.2} + a_{i.2} \qquad for \quad i =,2,3 \; and \; 4$$

Comparing the last two elements of the second row

$l_{2.1}u_{1.3} + l_{2.2}u_{2.3} = a_{2.3}$ and $l_{2.1}u_{1.4} + l_{2.2}u_{2.4} = a_{2.4}$ therefore

$$u_{2.i} = \frac{u_{2.1} - l_{2.1}u_{1.i}}{l_{2.2}} \quad For \quad i = 3,4$$

The comparison of the last two elements of the third row yields

$l_{3.1}u_{1.3} + l_{3.2}u_{2.3} + l_{3.3} = a_{3.3}$ and $l_{4.1}u_{1.3} + l_{4.2}u_{2.3} + l_{4.3} = a_{4.3}$

therefore $l_{i.3} = a_{i.3} - l_{i.1}u_{1.3} - l_{i.2}u_{2.3}$ $for\ i = 3.4$

The comparison of the last element of the third row yields

$$u_{3.4} = \frac{a_{3.4} - l_{3.1}u_{1.4} + l_{3.2}u_{2.4}}{l_{3.3}}$$

Similarly , the last element of the Fourth column.

yields $l_{4.4} = a_{4.4} - l_{4.1}u_{1.4} - l_{4.2}u_{2.4} - l_{4.3}u_{3.4}$

In general: for K=1,2,…,N. the elements of the decomposition matrices L and U of an NxN matrix A are given by:

$$u_{kk} = 1$$

$$l_{i.k} = a_{i.k} - \sum_{m=1}^{k-1} l_{i.m} U_{mk} \quad for\ i = k, k+1, \cdots, N$$

$$u_{kj} = \frac{1}{l_{kk}}\left[a_{kj} - \sum_{m=1}^{k-1} l_{km} u_{mj}\right] \quad for \quad j = k+1, k+2, \ldots, N$$

similarly, for K=1,2,…,N the elements of the decomposition matrices L and U of an NxN matrix A by **the Doolittle methods** are given by :

$$l_{kk} = 1$$

$$u_{kj} = a_{kj} - \sum_{m=1}^{k-1} l_{km} u_{mj} \quad for \quad j = k, k+1, \ldots, N$$

$$l_{ik} = \frac{1}{u_{kk}}\left[a_{ik} - \sum_{m=1}^{k-1} l_{im} l_{mk}\right] \quad for\ \ i = k, k+1, k+2, \ldots, N$$

The solution of the system of equations $Ly = b$ is given by

$$y_i = \frac{1}{l_{ii}}\left[b_i - \sum_{j=1}^{i-1} l_{ij}\, y_i \right] \qquad for\ i = 1,2,\ldots, N$$

and the solution of the system of equation $Ux = y$

is given by $$x_i = \frac{1}{u_{ii}}\left[y_i - \sum_{j=i+1}^{N} u_{ij}\, x_j \right] for\ i = N, N-1,\cdots,1$$

Example: Using the Crout factorization method, solve

$$4x_1 + 2x_2 + 3x_3 = 7$$
$$2x_1 - 4x_2 - x_3 = 1$$
$$-x_1 + x_2 + 4x_3 = -5$$

Solution: First let us find L and U such that

$$A=\begin{bmatrix} 4 & 2 & 3 \\ 2 & -4 & 1 \\ -1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} l_{1.1} & 0 & 0 \\ l_{2.1} & l_{2.2} & 0 \\ l_{3.1} & l_{3.2} & l_{3.3} \end{bmatrix} \begin{bmatrix} 1 & u_{1.2} & u_{1.3} \\ 0 & 1 & u_{2.3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{1.1} & l_{1.1}u_{1.2} & l_{1.1}u_{1.3} \\ l_{2.1} & l_{2.1}u_{1.2}+l_{2.2} & l_{2.1}u_{1.3}+l_{2.2}u_{2.3} \\ l_{3.1} & l_{3.1}u_{1.2}+l_{3.2} & l_{3.1}u_{1.3}+l_{3.2}u_{2.3}+l_{3.3} \end{bmatrix}$$

$$\therefore l_{1.1}=4, \quad l_{2.1}=2, \quad l_{3.1}=-1, \quad l_{1.1}u_{1.2}=2$$

$$\therefore u_{1.2}=\frac{2}{4}, \quad l_{1.1}u_{1.3}=3 \quad \therefore u_{1.3}=\frac{3}{4}, \quad l_{2.1}u_{1.3}+L_{2.2}u_{2.3}=-1$$

$$\Rightarrow 2u_{1.3}+l_{2.2}u_{2.3}=-1 \quad \Rightarrow 2\frac{3}{4}+l_{2.2}u_{2.3}=-1 \quad \Rightarrow l_{2.2}u_{2.3}=-1-\frac{3}{4}=\frac{-5}{2}$$

$$but\ l_{2.1}u_{1.2}+l_{2.2}=-4 \quad \Rightarrow +1+l_{2.2}=-4$$

$$\therefore l_{2.2} = -5 \Rightarrow \therefore -5u_{2.3} = \frac{-5}{2} \Rightarrow \therefore u_{2.3} = \frac{5}{10} = \frac{1}{2}$$

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 2 & -4 & -1 \\ -1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 2 & -5 & 0 \\ -1 & \frac{3}{2} & 4 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$$

we want to solve $Ax = b = L(Ux) \quad Let \quad Ux = y.$

Then we have

$$Ly = b \quad \Rightarrow \begin{bmatrix} 4 & 0 & 0 \\ 2 & -5 & 0 \\ -1 & \frac{3}{2} & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ -5 \end{bmatrix}$$

solving we get $\therefore y_1 = \frac{7}{4}$

$$2y_1 - 5y_2 = 1 \Rightarrow \therefore y_2 = \frac{(1 - 2y_1)}{-5} = \frac{1}{2},$$

$$-y_1 + \frac{2}{3}y_2 + 4y_3 = -5 \Rightarrow y_3 = -1$$

Now we have to solve $Ux = y$

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{7}{4} \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$\Rightarrow x_3 = -1 \quad , \Rightarrow x_2 + \frac{1}{2}x_3 = \frac{1}{2} \Rightarrow x_2 = 1$$

$$\Rightarrow x_1 + \frac{1}{2}x_2 + \frac{3}{4}x_3 = \frac{7}{4}$$

$$\Rightarrow x_1 + \frac{1}{2} - \frac{3}{4} = \frac{7}{4} \Rightarrow x_1 = \frac{5}{2} - \frac{1}{2} = \frac{4}{2} = 2 \Rightarrow x_1 = 2$$

Homework:

1)solve the linear system using Gaussian Elimination:

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ 2x_1 + 3x_2 - x_3 &= 4 \qquad , Ans: x_1 = 1,\ x_2 = 1,\ x_3 = 1 \\ -3x_1 + x_2 + x_3 &= 1 \end{aligned}$$

2)Solve the linear system using an LU decomposition:

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 10 \\ 2x_1 + x_2 - x_3 &= -1 \\ 6x_1 - 2x_2 + 2x_3 &= 22 \end{aligned}$$

Iterative Methods For Systems of Equations:

1) The Jacobi Method:

Consider a linear System $Ax = b$ given by

$$a_{1.1}x_1 + a_{1.2}x_2 + \cdots + a_{1.N}x_N = b_1$$

$$a_{2.1}x_1 + a_{2.2}x_2 + \cdots + a_{2.N}x_N = b_2$$

$$\vdots$$

$$a_{N.1}x_1 + a_{N.2}x_2 + \cdots + a_{N.N}x_N = b_N$$

Solve the first equation for x_1 the Second equation for x_2 and So forth. then

$$x_1 = \frac{1}{a_{1.1}}\left(b_1 - a_{1.2}x_2 - \cdots - a_{1.N}x_N\right)$$

$$x_2 = \frac{1}{a_{2.2}}\left(b_2 - a_{2.1}x_1 - \cdots - a_{2.N}x_N\right)$$

$$\vdots$$

$$x_N = \frac{1}{a_{N.N}}\left(b_N - a_{N.1}x_1 - \cdots - a_{N.N-1}x_{N-1}\right)$$

The System Can be Written as

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{i=1}^{N} a_{ij}\, x_j\right) for\, i = 1,2,3,\cdots, N \ldots(1)$$

Provided $a_{ii} \neq 0$

The entire Sequence of Jacobi iterates is defined from (1) as

$$x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{i=1}^{N} a_{ij}\, x_j^{k}\right) \text{for } i = 1,2,3,\cdots, N \quad and \quad k = 0,1,2.... \qquad (2)$$

equation (2) is easy to program for Computation

It is useful to Write equation (2) in matrix-Vector notation

To study the convergence of the Jacobi Method

$$Let\ A = \begin{pmatrix} a_{1.1} & \cdots & a_{1.N} \\ \vdots & \ddots & \vdots \\ a_{N.1} & \cdots & a_{N.N} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_{2.1} & 0 & \cdots & 0 \\ a_{N.1} & \cdots & & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} a_{1.1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_{N.N} \end{pmatrix} + \begin{pmatrix} 0 & a_{1.2} & a_{1.N} \\ 0 & 0 & a_{2.N} \\ 0 & 0 & 0 \end{pmatrix} = L + D + U$$

Where L.D. and U are the Strictly lower, diagonal
And Strictly upper triangular parts of A
The equation $Ax = b$, which is $(L + D + U)x = b$
Can be written as

$$Dx = -(L + U)x + b$$

This reduces to

$$x = -D^{-1}(L + U)x + D^{-1}b$$

And sequence of iterates is given by

$$x^{(k+1)} = -D^{-1}(L + U)x^{(k)} + D^{-1}b \;\; for \;\; K = 0,1........(3)$$

<u>EXAMPLE:</u>

$$Solve\, x_1 + 10\, x_2 = 11 \qquad 8\, x_1 + x_2 = 9$$

Using (1) Jacobi method

(2) Express the System in the forms of equation (3) ?

Solution:

Solving the first equation for x_1 and the Second for x_2 yields

$$x_1 = 11 - 10\ x_2$$
$$x_2 = 9 - 8\ x_1$$

(i) the Jacobi iterates

$$x_1^{(k+1)} = 11 - 10\ x_2^{(k)} \quad and \quad x_2^{(k+1)} = 9 - 8x_1^{(k)}, k = 0,1,...$$

$$Let\ x_1^{(0)} = 0\ and\ x_2^{(0)} = 0. then$$

$$We\ have$$

$$x_1^{(1)} = 11 - 10\ x_2^{(0)} = 11,\ for\ K = 1$$
$$x_2^{(1)} = 9 - 8\ x_1^{(0)} = 9,\ for\ K = 1$$

K	$x_1^{(k)}$	$x_2^{(k)}$
0	0	0
1	11	9
2	-79.0	-79.0
3	801.0	641.0
4	-6399	-6399
5	64.001	51.201

Not Converging

(ii) We have

$$A=\begin{pmatrix}1 & 10\\ 8 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 8 & 0\end{pmatrix}+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & 10\\ 0 & 0\end{pmatrix}=L+D+U$$

$$D^{-1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\Rightarrow x^{(k+1)}=-D^{-1}(L+U)x^{(k)}+D^{-1}b$$

$$i.e/$$

$$\begin{pmatrix}x_1\\ x_2\end{pmatrix}^{(k+1)}=-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 10\\ 8 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}^{k}+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}11\\ 9\end{pmatrix}=-\begin{pmatrix}0 & 10\\ 8 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}^{k}+\begin{pmatrix}11\\ 9\end{pmatrix}$$

Our Next question is to determine the Condition for Which the Sequence will Converges to the Solution of a system.

We need to know first of all:

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_N| = \sum_{i=1}^{N} |x_i|$$

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2} + \cdots + x_N^2 \sqrt{\sum_{i=1}^{N} |x_i|^2}$$

$$\|x\|_x = \max \{|x_1|, |x_2|, \cdots, |N|\} = \max |x_i|$$

$e.g$

$Find\ \|A\|_1, \|A\|_2\ and\ \|A\|_x\ if$

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{vmatrix}$$

Solution

$$\|A\|_1 = \max\{|1| + |3| + |5|, |2| + |4| + |6|, |3| + |5| + |7|\} = \max\{9, 12, 15\} = 15$$

Theorem:

Let A be a strictly diagonally dominant matrix.

Then the Jacobi and Gauss-Seidel iterations Converge to the unique solution of $Ax = b$ for any $x^{(0)}$

Definition:

A is S.D.D. if $$|a_i| \succ \sum_{\substack{j=1 \\ j \neq i}}^{N} |a_{i\,j}| \qquad i = 1,2,......, N$$

e.g/

$$A = \begin{pmatrix} 5 & -1 & -2 \\ 1 & -3 & 0 \\ 1 & 2 & 10 \end{pmatrix}$$

$$5 \succ |-1| + |-2| = 3 \quad , \quad -3 \succ |1| + |0| = 1 \; , \quad 10 \succ |1| + |2| = 3$$

Convergence of the Jacobi Methods:

for Jacobi to converge

$$\|B\| \le \|D^{-1}(L+U)\| \langle 1$$

$$Where\ B = -D^{-1}(L+U)$$

i.e the eignvalues $\lambda_i \prec 1$

and , always we have to assume value for the vector $x_i^{(0)}$ $for\ \ i = 1,2,\ldots\ldots.$

EXAMPLE: Consider the linear System:

$$
\begin{aligned}
-2x_1 + x_2 \qquad\qquad &= -2 \\
x_1 - 2x_2 + x_3 &= 0 \\
x_2 - 2x_3 &= -3
\end{aligned}
$$

Then

$$A=\begin{pmatrix}-2 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -2\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}+\begin{pmatrix}-2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -2\end{pmatrix}+\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$$

The Jacobi matrix is

$$B_j=-D^{-1}(L+U)=\begin{pmatrix}0 & \frac{1}{2} & 0\\ \frac{1}{2} & 0 & \frac{1}{2}\\ 0 & \frac{1}{2} & 0\end{pmatrix}$$

To find the eignvalues

$$Let\ \lambda I=\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}$$

$$\Rightarrow B_j - \lambda I = \begin{pmatrix} 0-\lambda & \frac{1}{2} & 0 \\ \frac{1}{2} & 0-\lambda & \frac{1}{2} \\ 0 & \frac{1}{2} & 0-\lambda \end{pmatrix}$$

$$\det\left(B_j - \lambda 1\right) = -\lambda\left[\lambda^2 - \frac{1}{4}\right] - \left[\frac{-\lambda}{4}\right]$$

$$= -\lambda^3 + \frac{\lambda}{4} + \frac{\lambda}{4} = -\lambda\left[\lambda^2 - \frac{1}{4}\right]$$

$$= -\lambda\left(\lambda^2 - \frac{1}{4}\right) = 0$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = -\sqrt{\frac{1}{2}}, \lambda_3 = +\sqrt{\frac{1}{2}}$$

$$i.e. \lambda_i \prec 1$$

$$x^{(K+1)} = -D\left(L+U\right)x^{(K)} + D^{-1}b$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^{(K+1)} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^{(K)} + \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix}$$

Assuming $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$

K	$X_1^{(K)}$	$X_2^{(K)}$	$X_3^{(K)}$
0	0	0	0
1	1	0	1.5
2	1	1.25	1.5
5	1.9375	1.875	2.4375
10	2.17188	2.42188	2.67188
30	2.24992	2.49992	2.74992

$$x_1 = \frac{9}{4}, x_2 = \frac{5}{2}, x_3 = \frac{11}{4}$$

this is the exact solution

To work an inverse matrix using ad-joint method:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 4 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

$$A^* = \begin{pmatrix} \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 4 & 0 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} \\ \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 3 & -4 & -1 \\ -12 & 2 & 4 \\ 7 & 0 & -7 \end{pmatrix}$$

$$AA^* = A^*A = -14I$$

$$A^{-1} = \frac{A^*}{\det A} = \begin{pmatrix} -\frac{3}{14} & \frac{4}{14} & \frac{1}{14} \\ \frac{12}{14} & -\frac{2}{14} & -\frac{4}{14} \\ -\frac{7}{14} & 0 & \frac{7}{14} \end{pmatrix}$$

2) Gauss- Sidel Method:

We rewrite equation (1) as

$$x_i = \frac{1}{a_{i\,i}}\left[b_i - \sum_{j=1}^{i-1} a_{i\,j} x_j - \sum_{j=i+1}^{N} a_{i\,j}.x_j \right] \cdots (4)\ for\ i = 1,2,..........N$$

provided $a_{i\,i} \neq 0$

from the above equation (4) the Gauss-Seidel iteration Sequence can be defined as

$$x_i^{(K+1)} = \frac{1}{a_{i\,i}}\left[b_{i\quad i} - \sum_{j=1}^{i-1} a_{i\,j} x_j^{(K+1)} - \sum_{j=i+1}^{N} a_{i\,j} x_j^{(K)} \right] \cdots (5)$$

$$for\ i = 1,2,..........N \quad and \quad k = 0,1,2,...$$

$$and\ \text{given}\ \ x_i^{(0)}$$

Each updated component $x_i^{(K+1)}$
Is used in the calculation of the next Component and therefore, for computer calculation, the new value can be immediately stored at the location where the old value was stored this reduces the number of necessary locations

Equation (5) Can be written as

$$a_{11}x_1^{(K+1)} = b_1 - a_{1.2}x_2^{(K)} - \ldots\ldots - a_{1.N}\ x_N^{(K)}$$

$$a_{2.1}x_1^{(K+1)} + a_{2.2}x_2^{(K+1)} = b_2 - a_{2.3}x_3^{(K)} - a_{2.N}x_N^{(K)}$$

$$\vdots$$

$$a_{N.1}x_1^{(K+1)} + a_{N.2}x_2^{(K+1)} + \ldots\ldots\ldots + a_{N.N}x_N^{(K+1)} = b_N$$

In matrix notation:

$$\begin{pmatrix} a_{1.1} & 0 & \cdots & 0 \\ a_{2.1} & a_{2.2} & \cdots & 0 \\ a_{N.1} & a_{N.2} & \cdots & a_{N.N} \end{pmatrix} \begin{bmatrix} x_1^{(K+1)} \\ x_2^{(K+1)} \\ \vdots \\ x_N^{(K+1)} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} - \begin{pmatrix} 0 & a_{1.2} & \cdots & a_{1.N} \\ 0 & 0 & & a_{2.N} \\ 0 & 0 & & 0 \end{pmatrix} \begin{bmatrix} x_1^{(K)} \\ x_2^{(K)} \\ \vdots \\ x_N^{(K)} \end{bmatrix}$$

$$(D+L)x^{(K+1)} = b - U\,x^{(k)}$$

$$i.e. \quad x^{(K+1)} = -(D+L)^{-1}U\,x^{(k)} + (D+L)^{-1}b.......*$$

EXAMPLE: solve

$$x_1 + 10x_2 = 11$$

$$8x_1 + x_2 = 9$$

Using

(i) the Gauss-Seidel Method

(ii) Express the System in the form of * equation

Solution:

The Gauss Seidel iterations are given by

$$x_1^{(K+1)} = 11 - 10x_2^{(K)} \quad for\ K = 0,1,\ldots\ldots\ldots$$

$$x_2^{(K+1)} = 9 - 8x_1^{(K+1)} \quad for\ K = 0,1,\ldots\ldots\ldots$$

$$\mathbf{Let}\ x_1^{(0)} = x_2^{(0)} = 0$$

Then for K=0

$$x_1^{(1)} = 11 - 10x_2^{(0)} = 11$$

$$x_2^{(1)} = 9 - 8x_1^{(1)} = 9 - 88 = -79$$

K	$X_1^{(k)}$	$X_2^{(K)}$
0	0	0
1	11	-79
2	801	-6399
3	64.001	-511.99

It is diverging.

Convergence of the Gauss-Seidel Method

For Convergence

$$\|B\| = (D+L)^{-1}U\| \prec 1$$

$$i.e.\ The\ eiganvalues\ \lambda_i \prec 1.$$

$$Where\ B = -(D+L)^{-1}U.$$

We always have to assume values for $x_2^{(0)}$ $for\ i = 1,2,.......$

EXAMPLE: Consider the linear System.

$$
\begin{aligned}
-2x_1 + x_2 \quad\quad &= -2 \\
x_1 - 2x_2 + x_3 &= 0 \\
x_2 - 2x_3 &= -3
\end{aligned}
$$

Solution in this case

$$
A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} = \underset{L}{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} + \underset{D}{\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}} + \underset{U}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}
$$

The Gauss-Seidel Matrix is

$$B = -(D+L)^{-1}U = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{8} & \frac{1}{4} \end{pmatrix}$$

$$Let \qquad \lambda I = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \Rightarrow B - \lambda I = \begin{pmatrix} -\lambda & \frac{1}{2} & 0 \\ 0 & \frac{1}{4}-\lambda & \frac{1}{2} \\ 0 & \frac{1}{8} & \frac{1}{4}-\lambda \end{pmatrix}$$

$$\det(B-\lambda I) = -\lambda\left[\left(\frac{1}{4}-\lambda\right)^2 \frac{1}{16}\right] = -\lambda\left[\lambda^2 - \frac{2}{4}\lambda + \frac{1}{16} - \frac{1}{16}\right]$$

$$= \lambda^2\left(\lambda - \frac{1}{2}\right)$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = \frac{1}{2}$$

$$i.e.\ \lambda_i \succ 1$$

It Converges

$$x^{(K+1)} = -(D+L)^{-1}Ux^{(K)} + (D+L)^{-1}b$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^{(K+1)} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{8} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^{(K)} + \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{2} & 0 \\ -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ -3 \end{pmatrix}$$

Assuming $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$

K	$x_1^{(K)}$	$x_2^{(K)}$	$x_3^{(K)}$
0	0	0	0
1	1	0.5	1.75
2	1.25	1.5	2.25
5	2.125	2.375	2.6875
10	2.24609	2.49609	2.74805

Approximation Theory:

Discrete Least – Squares approximation :

So far we discussed the techniques to compute x of a given linear System $Ax = b$ where A is a Square matrix. If A is nonsingular, then there exists a unique solution. In this section, we turn our attention to a system of m equations in n unknowns where $m \neq n$

Thus if A has m rows and n columns, then x is a vector with n components and b is a vector with m components. If m>n. then we have more equations than unknowns. Such systems are usually over determined.

Over determined systems do arise in practice and need to be solved.

Let us have

$$a_0 + 11a_1 = 20 \quad , \quad a_0 + 19a_1 = 26$$

$$a_0 + 13a_1 = 21 \quad , \quad a_0 + 23a_1 = 32$$

$$a_0 + 17a_1 = 24 \quad , \quad a_0 + 27a_1 = 34$$

$$i.e. \begin{pmatrix} 1 & 11 \\ 1 & 13 \\ 1 & 17 \\ 1 & 19 \\ 1 & 23 \\ 1 & 27 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 20 \\ 21 \\ 24 \\ 26 \\ 32 \\ 34 \end{pmatrix} \ldots\ldots\ldots\ldots\ldots(1)$$

One possibility is to determine a_0 , and a_1 from apart of equation (1) by ignoring the rest. However Since the data comes from the same source. It is difficult to know which equations contain large errors.

Thus we can not Justify determining a_0 and a_1 from apart of equation (1) by ignoring the rest.

It seems reasonable to choose a_0 and a_1 Such that the average error in these six equations is minimum.

There are many ways to define this average error.

But the most convenient and often used is the sum of squares.

$$E^2 = (20 - a_0 - 11a_1)^2 + (21 - a_0 - 13a_1)^2 + (24 - a_0 - 17a_1)^2$$
$$+ (26 - a_0 - 1a_1)^2 + (32 - a_0 - 23a_1)^2 + (34 + a_0 - 27a_1)^2$$

Consider a System

$$a_{1.1}x_1 + a_{1.2}x_2 + \dots\dots\dots + a_{1.n}x_n = b_1$$
$$a_{2.1}x_1 + a_{2.2}x_2 + \dots\dots\dots + a_{2.n}x_n = b_2$$
$$\vdots$$
$$a_{m.1}x_1 + a_{m.2}x_2 + \dots\dots\dots + a_{m.n}x_n = b_{m.}$$

Where A is mxn and m>n. define the vector

$$r = b - Ax.$$
$$= \left[b_1 - \sum_{i=1}^{n} a_{1i}x_i , b_2 - \sum_{i=1}^{n} a_{2i}x_i ,\dots\dots\dots\dots, b_m - \sum_{i=1}^{n} a_{mi}\, x_i \right]^{\backslash}$$
$$= [r_1, r_2, \dots\dots, r_m]^{\backslash}$$

Then

$$E^2 = r_1^2 + r_2^2 + \dots\dots\dots + r_m^2$$
$$E^2 = r^{\backslash} r$$

$$E^2 = (b - Ax)^{\backslash} \quad (b - Ax)$$

$$= (b_1 - a_{11}\ x_1 - a_{12}\ x_2 - \cdots - a_{1n}\ x_n)^2$$

$$+ (b_2 - a_{21}\ x_1 - a_{22}\ x_2 - \cdots - a_{2n}\ \ x_n)^2$$

$$+ \cdots + (b_m - a_{m1}\ x_1 - a_{m2}\ x_2 - \cdots - a_{mn}\ x_n)^2$$

We wish to find $x \in R^n$ for which E^2 is minimum

Which is called a <u>least square solution of</u> $Ax = b.$ E^2 is min where

$$\frac{\partial}{\partial x_1}\left(\boldsymbol{r}^{\backslash}\ r\right) = \frac{\partial}{\partial x_2}\left(\boldsymbol{r}^{\backslash}\ r\right) = \cdots \frac{\partial}{\partial x_n}\left(\boldsymbol{r}^{\backslash}\ r\right) = 0 \rightarrow (1)$$

$$Since\ \boldsymbol{r}^{\backslash} r = r_1^2 + r_2^2 + \cdots + r_m^2$$

$$\frac{\partial}{\partial x_1}\left(\boldsymbol{r}^{\backslash}\ r\right) = \frac{\partial}{\partial x_1}\left(r_1^2 + r_2^2 + \cdots + r_m^2\right)$$

$$= 2r_1\ \frac{\partial r_1}{\partial x_1} + 2r_2\ \frac{\partial r_2}{\partial x_1} + \cdots + 2r_m\ \frac{\partial f_m}{r x_1}$$

$$= -2\ r_1\ \ a_{11} - 2r_2\ \ a_{21} \cdots - 2r_m\ a_{m1}$$

It can be shown that $$\frac{\partial}{\partial x_j}\left(r'r\right) = -2\sum_{i=1}^{m} r_i \ a_{i\,j} \quad \rightarrow (2)$$

Substituting equation (2) in (1) gives

$$\sum_{i=1}^{m} r_i \ a_{i\,j} = \sum_{i=1}^{m} a_{ij} r_i = 0 \quad for \ \ j = 1, 2, \cdots, n$$

In other words $$\begin{bmatrix} a_{1.1} & a_{1.2} & \cdots & a_{m.1} \\ a_{2.1} & a_{2.2} & \cdots & a_{m.2} \\ \vdots & & & \\ a_{1.x} & a_{2.n} & \cdots & a_{m.n} \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The system of equation is $$A^{\backslash} r = 0 \quad (3)$$

Substituting $r = b - Ax$ in equation (3) we get

$$A'\left(b - Ax\right) = 0$$

$$\therefore A' \ Ax = A'b$$

<u>which is called a normal equation</u>

Many times an over determined system arises when we try to find a_0 and a1 such that $y=a_0 + a_1 x$ is the least squares to fit to the of data given in table.

$$x = x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_N$$

$$y = y_1 \quad y_2 \quad y_3 \quad \cdots \quad y_N$$

For each pair (x_i, y_i) the equation

$y_i = a_0 + a_1 x_i$ should hold.

therefore

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \\ 1 & x_N \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

The normal equation: $A'Ax = A'b$ reduces to

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & & x_N \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \\ 1 & x_N \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & & x_N \end{bmatrix} \begin{bmatrix} y_1 \\ y_N \end{bmatrix}$$

Which can be simplifies to

$$\begin{bmatrix} N & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i & \sum_{i=}^{N} x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} y_i \\ \sum_{i=1}^{N} x_i \; y_i \end{bmatrix} \rightarrow (4)$$

A`A in equation (4) is symmetric solving equation (4) for a_0 and a_1

We get

$$a_0 = \frac{\sum_{i=1}^{N} y_i \sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i \; y_i\right)\left(\sum_{i=1}^{N} x_i\right)}{N\left(\sum_{i=1}^{N} x_i^2\right) - \left(\sum_{i=1}^{N} x_i\right)^2} \quad *$$

$$a_1 = \frac{N\left(\sum_{i=1}^{N} x_i \; y_i\right) - \left(\sum_{i=1}^{N} y_i\right)\left(\sum_{i=1}^{N} x_i\right)}{N\left(\sum_{i=1}^{N} x_i^2\right) - \left(\sum_{i=1}^{N} x_i\right)^2} \quad *$$

Example:

Using the least squares method find the linear polynomial that fits the following data:

x_i	-1	0	1
y_i	-3	-1	2

Solution: $y = a_0 + a_1 x$

$$\begin{bmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

$$3a_0 = -2 \qquad \therefore a_0 = \frac{-2}{3} \qquad , \quad 2a_1 = 5 \qquad \therefore a_1 = 5/2$$

$$\therefore y = \frac{-2}{3} + \frac{5}{2} x$$

Example:

The experimental data points given below indicate a curve having the form :

$$y = \frac{ax}{b + x}$$

x_i	1	2	3
y_i	1	1.333	1.5

Determine the least square fit of this of function to the data ?

Solution:

$$y = \frac{a\,x}{b+x}$$

$$y\,b + y\,x = a\,x$$

$$\therefore\ y\,x = a\,x - b\,y$$

$$y = a - \frac{by}{x}$$

Let $Z = \frac{y}{x}$, $\therefore\ y = a - b\,z$ $\qquad a_0 = a \qquad a_1 = -b$

$$\begin{bmatrix} n & \sum z_i \\ \sum z_i & \sum z_i^2 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum z_i y_i \end{bmatrix}, \quad \begin{bmatrix} 3 & 2.1665 \\ 2.1665 & 1.6942 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3.8333 \\ 2.6384 \end{bmatrix}$$

$$\therefore\ a_0 \approx 2 = a \qquad a_1 = -1 = -b$$

Example: The experimental data points given below indicate a curve having the form:

$$y = \frac{x^2}{a} e^x + x e^b$$

x_i	1	2	3
y_i	2.102	8.223	35.074

Determine the least square fit of this function to the data?

Solution:

$$\frac{y}{x} = \frac{1}{a} x e^x + e^b \qquad let \quad Z = \frac{y}{x}$$

$$Z = \frac{1}{a} w + e^b \qquad where \quad w = x e^x$$

$$Z_i = a_0 + a_1 w_i \qquad where\ a_0 = e^b, a_1 = \frac{1}{a}$$

x_i	y_i	Z_i	W_i	W_i^2	$W_i\ Z_i$
1	2.102	2.102	2.718	7388	5.7132
2	8.223	4.112	14.778	218.389	60.767
3	35.074	11.691	60.257	3630.906	704.965

$$\begin{bmatrix} N & \sum w_i \\ \sum w_i & \sum w_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum z_i \\ \sum z_i w_i \end{bmatrix}$$

$$\begin{bmatrix} N & 77.753 \\ 77.753 & 3856.683 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 17.905 \\ 770.945 \end{bmatrix}$$

$$b = \ln a_0 = \ln(1.649) = 0.5 \qquad \text{, and} \qquad a = \frac{1}{a_1} = \frac{1}{0.167}\ 6.00$$

We could use a polynomial of degree M given by

$$P_M(x) = a_1 + a_1 x + \cdots + a_M \; x^M \qquad \cdots(1)$$

Where the coefficient $a_0, a_1, \ldots\ldots, a_M$

are to be determined to fit a given set of data points

$$(x_1, y_1), (x_2, y_2), \cdots, (x_N, y_N)$$

Since each pair (x_i, y_i) satisfies equation (1) we get

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^M \\ 1 & x_2 & x_2^2 & \cdots & x_2^M \\ 1 & x_N & x_N^2 & \cdots & x_N^M \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_M \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \qquad \ldots\ldots(2)$$

The least squares solution of (2) is given by

$$A'Aa = A'b \quad * \quad , \qquad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

Example:

Fit the data below with the least squares polynomial of degree two

x_i	0	0.25	0.5	0.75	1.00
y_i	1	1.284	1.6487	2.1170	2.7183

Solution:

$$A'Aa = A'b$$

the polynomial is

$$y = a_0 + a_1 x + a_2 x^2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0.25 & 0.5 & 0.75 & 1 \\ 0 & 0.0625 & 0.25 & 0.5625 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.25 & 0.0625 \\ 1 & 0.5 & 0.25 \\ 1 & 0.75 & 0.5625 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0.25 & 0.5 & 0.75 & 1 \\ 0 & 0.0625 & 0.25 & 0.5625 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1.284 \\ 1.6487 \\ 2.117 \\ 2.7183 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 5 & 2.5 & 1.875 \\ 2.5 & 1.875 & 1.5625 \\ 1.875 & 1.5625 & 1.3828 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 8.768 \\ 5.4514 \\ 4.4015 \end{bmatrix}$$

$$5a_0 \quad +2.5a_1 \quad +1.875a_2 = 8.768$$
$$2.5a_0 \quad +1.875a_1 \quad +1.5625a_2 = 5.4514$$
$$1.875a_0 +1.5625a_1 +1.3828a_2 = 4.4015$$

$$\therefore \quad a_0 = 1.0052, \quad a_1 = 0.8641, \quad a_2 = 0.8437$$

The polynomials is $y = a_0 + a_1 x + a_2 x^2$

$$= 1.0052 + 0.8641x + 0.8437x^2$$

Home work:

1) using the least squares method, find the linear polynomials that fits the following data.

x_i	2	3	4
y_i	2	$9/4$	$7/4$

2) Using the least squares method, Find the quadratic polynomial that fits the following data :

x_i	1	2	3	4
y_i	1	2	4	9

Orthogonal polynomials and least squares approximation

suppose $f \in C[a,b]$

and that polynomial of degree at most n, P_n is required that will minimize

$$\int_a^b \left[f(x) - P_n(x)\right]^2 dx \qquad \cdots(1)$$

To determine a least squares approximating polynomial that is, a polynomial to minimize expression (1) let

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = \sum_{k=0}^{n} a_k x^k$$

$$\text{and define} \quad E(a_0, a_1, \cdots, a_n) = \int_a^b \left[f(x) - \sum_{k=0}^{n} a_k x^k \right]^2 dx$$

The problem is to find real coefficients $a_0, \ldots, a_n$ that

Will minimize E from the calculus of functions of several variables, a necessary condition for the number $a_0, \cdots, a_n$

to minimize E is that

$$\frac{\partial E}{\partial a_j} = 0 \qquad for \quad j = 0, 1, \cdots, n$$

$$\sin ce \quad E = \int_a^b [f(x)]^2 \, dx - 2\sum_{k=0}^{n} a_k \int_a^b x^k f(x)\, dx + \int_a^b \left[\sum_{k=0}^{n} a_k x^k\right]^2 dx$$

$$\frac{\partial E}{\partial a_j} = -2\int_a^b x^j f(x) dx + 2\sum_{k=0}^{n} a_k \int_a^b x^{j+k} dx \quad j = 0, 1, \cdots, n$$

Hence in order to find P_n ,the linear equations $(n+1)$

$$\sum_{k=0}^{n} a_k \int_a^b x^{j+k} dx = \int_a^b x^j f(x) dx \ldots j = 0,1,2,\ldots,n$$

Must be solved for the (n+1) unknowns $a_j, \quad j = 0,1,\cdots,n$

These equations are called the normal equations it can be shown that the normal equations always have a unique solution provided

$$f \in C[a,b] \;\; \textit{and} \;\; \mathrm{a} \neq \mathrm{b}$$

Example: Find the least – squares approximating polynomial of degree two for the function $f(x) = \sin \pi x$ on the interval $[0,1]$

Solution: The normal equations for $P_2(x) = a_0 + a_1 x + a_2 x^2$

$$a_0 \int_0^1 1 dx + a_1 \int_0^1 x\, dx + a_2 \int_0^1 x^2\, dx = \int_0^1 Sin\, \pi\, x\, dx$$

$$a_0 \int_0^1 x\, dx + a_1 \int_0^1 x^2\, dx + a_2 \int_0^1 x^3\, dx = \int_0^1 x\, Sin\, \pi\, x\, dx$$

$$a_0 \int_0^1 x^2\, dx + a_1 \int_0^1 x^3\, dx + a_2 \int_0^1 x^4\, dx = \int_0^1 x^2\, Sin\, \pi\, x\, dx$$

Performing the integrations yields

$$a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 = \frac{2}{\pi}$$

$$\frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 = \frac{1}{\pi}$$

$$\frac{1}{3}a_0 + \frac{1}{4}a_1 + \frac{1}{5}a_2 = \frac{\pi^2 - 4}{\pi^3}$$

Solving the equations together to obtain

$$a_0 = \frac{12\pi^2 - 120}{\pi^3} \approx -0.50465 \quad , a_1 = \frac{720 - 60\pi^2}{\pi^3} \approx 4.12251$$

$$a_2 = \frac{60\pi^2 - 720}{\pi^3} \approx -4.12251$$

Consequently, the least squares polynomial approximation of degree two for

$$f(x) = Sin\,\pi x \quad on[0,1] \qquad \text{is}$$

$$p_2(x) = -4.12251x^2 + 4.12251x - 0.50465$$

Eigen values and eigenvectors: The power method:

Let A be an NxN matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N$

such that $|\lambda_1| \succ |\lambda_2| \geq |\lambda_3| \cdots \geq |\lambda_N|$

Assume that A has N linearly independent eigenvectors $V_1, V_2, \cdots, V_N$

associated with each of these eigenvalues since $\{V_1, V_2, \cdots, V_N\}$

form a basis of R^N, we can express any given vector $x^{(0)}$ as

$$x^{(0)} = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_N v_N$$

Where

$\alpha_1, \alpha_2, \cdots, \alpha_N$ are constants. Multiplying both sides of the equation by A gives

$$Ax^{(0)} = \alpha_1 Av_1 + \alpha_2 Av_2 + \cdots + \alpha_N Av_N$$
$$= \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 + \cdots + \alpha_N \lambda_N v_N$$

Inductively, for any positive integer K:

$$A^k x^{(0)} = \alpha_1 \lambda_1^{(k)} v_1 + \alpha_2 \lambda_2^k v_2 + \cdots + \alpha_N \lambda_N^k v_N$$
$$= \lambda_1^{(k)} \left\{ \alpha_1 v_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1} \right)^k v_2 + \cdots + \alpha_N \left(\frac{\lambda_N}{\lambda_1} \right)^k v_N \right\}$$

Since

$$\left| \lambda_i / \lambda_1 \right| \prec 1 \quad for \quad i \geq 2 \quad then$$

$$A^k x^{(0)} \to \lambda_1^k \alpha_1 v_1 \; as \, K \to \infty$$

For any given vector $x^{(0)}$ we generate the sequence given by

$$x^{(k)} = A x^{(k-1)} \quad for\ K = 1,2,\cdots$$

We can verify that $x^{(k)} = A^{(k)} x^{(0)} \quad for\ K = 1,2,\cdots \qquad (1)$

Thus $x^{(k)} \to \lambda_1^k \alpha_1 v_1 \quad as\ k \to \infty$

Since the sequence in equation (1) converges to zero if $|\lambda_1| \prec 1$ and diverges if $|\lambda_1| \succ 1$

Equation (1) may not be a practical sequence to compute the $x^{(k)}$

dominant eigenvalue. $x^{(k)}$

It is desirable to keep $x^{(k)}$ within computational limits by scaling

This can be done by dividing $\left\| x^{(k)} \right\|_x$ by its absolute largest

at each step. Let $x^{(0)}$ be an initial guess. Then define

component which is denoted by

$$Z^{(0)} = \frac{x^{(0)}}{\left\| x^{(0)} \right\|_x}$$ compute $x^{(1)} = A\,z^{(0)}$ Then

define $Z^{(1)} = \frac{x^{(1)}}{\left\| x^{(1)} \right\|_x}$ and continue we obtain

$$\begin{cases} Z^{(k)} = x^{(k)} \Big/ \left\| x^{(k)} \right\|_x \\ x^{(k+.)} = A\,z^{(k)} \end{cases} \qquad \text{for} \quad k = 0,1,\cdots \qquad \ldots(2)$$

Since $V_1, V_2, \cdots, V_N$ are linearly independent eignvectors we express

$Z^{(0)}$ as $Z^{(0)} = \beta_1\, v_1 + \beta_2\, v_2 + \cdots + \beta_N\, v_N$

Where $\beta_1, \beta_2, \cdots, \beta_N$ are constants , we compute

$$x^{(1)} = A\,z^{(0)} \quad \text{and} \quad z^{(1)} = x^{(1)} \Big/ \left\| x^{(1)} \right\|_x = A\,z^{(0)} \Big/ \left\| A\,z^{(0)} \right\|_x$$

Similarly

$$Z^{(k)} = \frac{A^k\ z^{(0)}}{\left\|A^k\ Z^{(0)}\right\|_x} = \frac{\lambda_1^k}{\left\|A^k\ Z^{(0)}\right\|_x}\left\{\beta_1\ v_1 + \beta_2\left(\frac{\lambda_2}{\lambda_1}\right)^k v_2 + \cdots + \beta_N\left(\frac{\lambda_N}{\lambda_1}\right)^k v_N\right\}$$

$$as\ k \rightarrow \infty \quad \cdots(3)$$

$$Z^{(k)} \rightarrow \frac{\beta_1\, v_1}{\left\|A^k\, Z^{(0)}\right\|_x}\ \lambda_1^k \cdots (4)$$

Multiplying equation (3) by A yields

$$x^{(k+1)} = A\,z^{(k)}$$

$$= \frac{\lambda_1^{(K+1)}}{\left\|A^k\ Z^{(0)}\right\|_x}\left\{\beta_1\ v_1 + \beta_2\left(\frac{\lambda_2}{\lambda_1}\right)^{k+1} v_2 + \cdots + \beta_N\left(\frac{\lambda_N}{\lambda_1}\right)^{k+1} v_N\right\}$$

$$As\ k \rightarrow \infty$$

$$x^{(k+1)} \rightarrow \frac{\beta_1\ v_1}{\left\|A^k\ Z^{(0)}\right\|_x}\lambda_1^{k+1} = \lambda_1\left(\frac{\beta_1\ v_1}{\left\|A^k\ Z^{(0)}\right\|_x}\right)\lambda_1^k \cdots\cdots(5)$$

Using equation (4) in (5) we get $x^{(k+1)} \to \lambda_1 Z^{(k)}$

for large K, we have $x^{(k+1)} = A Z^{(k)} = \lambda_1 Z^{(k)}$

Multiplying by $\left(Z^{(k)}\right)^{\backslash}$ we get

$$\lambda_1 = \frac{\left(z^{(k)}\right)' \left(A z^{(k)}\right)}{\left(z^{(k)}\right)' z^{(k)}}$$

Denoting

$$\lambda_1^{(k)} = \frac{\left(z^{(k)}\right)' \left(A z^{(k)}\right)}{\left(z^{(k)}\right)' z^{(k)}} \quad for \quad k = 0,1,\cdots \qquad (6)$$

We have

$$\lambda_1^{(k)} \to \lambda_1 \ as \ k \to \infty$$

It follows that the convergence of $Z^{(k)}$ to scalar multiple of v_1

Depends upon how fast the ratios

$(\lambda_i / \lambda_1)^k$ *for* $i = 2,3,\cdots, N$ *go to zero.*

we combin equ (2) *and* (6) *to get*

$$\left.\begin{aligned} &1)\ z^{(k)} = x^{(k)} \big/ \left\| x^{(k)} \right\|_x \\ &2)\ x^{(k+1)} = A z^{(k)} \\ &3)\ \lambda_1^{(k+1)} = \left(z^{(k)}\right)^1 x^{(k+1)} \big/ \left(z^{(k)}\right) z^{(k)} \end{aligned}\right\} (7) \quad \text{for} \quad k = 0,1,\ldots\ldots.$$

then as $k \to \infty, \quad \lambda_1^{(k+1)} \to \lambda_1$

and $z^{(k)} \to v_1$